Computing Conformal Maps onto Circular Domains
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Outline

1 Motivation
   - Multiply connected domains

2 Koebe’s construction

3 Computing the error in the Koebe construction
   - Some important constants
Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. As usual, topologize this with the basis consisting of all open rational disks as well as all sets of the form
Motivation
Koebe's construction
Computing the error in the Koebe construction

Multiply connected domains

Circular domains
Definition

1. A *domain* is an open connected subset of $\hat{\mathbb{C}}$.
2. A domain is *n*-connected if its complement has exactly $n$ components.
3. A domain is *degenerate* if a component of its complement consists of a single point.
4. A domain is *circular* if every component of its complement is a closed disk.
Let $\mathbb{D}$ be the unit disk centered at the origin.

**Example**

- $\mathbb{D}$ is 1-connected. So is $\hat{\mathbb{C}} - \overline{\mathbb{D}}$.
- Every annulus is 2-connected.

**Theorem (Riemann Mapping Theorem)**

*Every non-degenerate 1-connected domain is conformally equivalent to $\mathbb{D}$.*
**Question**

*Are all non-degenerate n-connected domains conformally equivalent?*

**Answer** = NO!
Motivation
Koebe’s construction
Computing the error in the Koebe construction

NOT EQUIVALENT!!!
This leads to the discussion of *canonical domain classes*.

**Definition**

A *canonical domain class* is a set of finitely connected domains $S$ such that each finitely connected non-degenerate domain is conformally equivalent to a domain in $S$.

There are as many of these as there are grains of sand...
Motivation
Koebe’s construction
Computing the error in the Koebe construction

Multiply connected domains

- Slit domains
  - Slit disk domains
  - Slit annulus domains
  - Circular slit domains
  - Parallel slit domains
  - Radial slit domains
- Circular domains
- Polygonal domains
- Polyarc domains
- etc.
Some problems from Pour-El and Richards:

1. What is the effective content of the Riemann Mapping Theorem?

2. What is the effective content of conformal mapping for multiply connected domains?
Some related work:

- Schwarz-Christoffel formula for mapping $\mathbb{D}$ onto polygonal domains.
- Schiffer, 1950: constructed slit domain maps from the Green’s function for the input domain.
- Hertling, 1999: for every non-degenerate 1-connected domain, $D$, a conformal map $f$ of $D$ onto $\mathbb{D}$ can be uniformly computed from $D$ and its boundary. This means:
there is a Turing machine that given

- a list of all basic sets whose closures are contained in a non-degenerate 1-connected domain $D$,
- a list of all basic sets which hit $\partial D$

enumerates all pairs $(S_1, S_2)$ such that

- $S_1, S_2$ are basic
- $\overline{S_1} \subseteq D$
- $f[\overline{S_1}] \subseteq S_2$. 
Recent importance of circular domains:
- Explicit formulas for Green’s functions
- Applications to aircraft design
- Heat-transfer equations
- Canonical domains in recent research in conformal mapping onto polygonal domains
Every non-degenerate $n$-connected domain is conformally equivalent to infinitely many circular domains. However, if we normalize the map we get uniqueness:

**Theorem (Koebe [7])**

*For every non-degenerate $n$-connected domain $D$ that contains $\infty$, there is a unique circular domain $C_D$ and a unique conformal map $f_D$ of $D$ onto $C_D$ such that $f_D(z) = z + O(z^{-1})$.*

It follows from Hertling’s theorem that when $n = 1$, $(D, \partial D) \mapsto (f_D, \partial C_D)$ is computable.
Motivation
Koebe's construction
Computing the error in the Koebe construction

D_2
D_1
D_3

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Circular domains
Motivation

Koebe's construction

Computing the error in the Koebe construction

\[ f_1(z) = z + O(z^{-1}) \]

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Circular domains
Motivation

Koebe’s construction

Computing the error in the Koebe construction

Circular domains

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Motivation
Koebe’s construction
Computing the error in the Koebe construction

$D_{2,0}, D_{2,1}, D_{2,2}$

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Circular domains
Motivation
Koebe’s construction
Computing the error in the Koebe construction

\[ f_3(z) = z + O(z^{-1}) \]
Motivation
Koebe’s construction
Computing the error in the Koebe construction

Circular domains
Motivation
Koebe's construction
Computing the error in the Koebe construction

\[ f_4(z) = z + O(z^{-1}) \]
... after infinitely many iterations ...
Motivation
Koebe’s construction
Computing the error in the Koebe construction

Circular domains
Let

\[ g_1 = f_1 \]
\[ g_{k+1} = f_{k+1} \circ f_k \ldots \circ f_1 \]

Then, \( g = \lim_{k \to \infty} g_k \) exists and is a conformal mapping of \( D \) onto \( C_D \) of the form \( z + O(z^{-1}) \). *n.b.* Known proofs of this use fact that \( f_D \) exists!
Throughout the rest of this talk, let $D$ range over finitely-connected non-degenerate domains only.

**Theorem (Main result)**

$$(D, \partial D, n) \mapsto (f_D, C_D, \partial C_D)$$ is computable.

Proof comes down to computing the error in the Koebe construction *from D and \partial D* So, let’s sketch how to do it.
Motivation

Koebe’s construction

Computing the error in the Koebe construction

Some important constants

\[ \rho_c \]

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Circular domains
Can transfer this back to any $D$ by setting $\rho_D = \rho_{C_D}$.
Motivation
Koebe’s construction
Computing the error in the Koebe construction

Some important constants

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Circular domains
Motivation

Koebe’s construction

Computing the error in the Koebe construction

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\[ \mu_C = (\text{least } \epsilon \text{ such that two circles touch})^{-1}. \]

Let \( \mu_D = \mu_C \).
Motivation
Koebe’s construction
Computing the error in the Koebe construction

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Circular domains
Motivation
Koebe's construction
Computing the error in the Koebe construction

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\[ C_1, C_2, C_3 \]
Let $\delta_C = \min_{i,j} \delta_{C}^{i,j}$.

Again, we can transfer this to any domain $D$ by setting $\delta_D = \delta_{CD}$.
Notation

Let

\[ \gamma_D = \frac{2\rho_D^2}{\pi \delta_D} \left[ \frac{2[\pi \mu_D^{-1}]^2}{\ln \mu_D^{-1}} + 1 \right] . \]

Theorem (Henrici, 1986 [5])

For all \( z \in D - \{\infty\} \) and all \( j \in \mathbb{N} \), \( |g_j(z) - g(z)| \leq \gamma \mu^{4|j/n|} \).

Key point: Proof assumes \( f_D \) and \( C_D \) exist! So, it does NOT provide a constructive proof of the existence of \( f_D \). All proofs of the convergence of the Koebe construction possess this feature!
To use this result we need to:

- Compute an upper bound on $\rho_D$.
- Estimate $\mu_D$ from above but below 1.
- Estimate $\delta_D$ from below.
Proposition

*From* \((D, \partial D, n)\), *we can compute an upper bound on* \(\rho_D\).*
Motivation

Koebe’s construction
Computing the error in the Koebe construction

Notation

Let $r_{\text{min}}(D)$ be the smallest radius of a circle in $C_D$.
Let $d_{\text{min}}(D)$ be the smallest distance between two circles in $C_D$.

Proposition

$$\delta_D \geq \frac{1}{\frac{1}{r_{\text{min}}(D)} + \frac{1}{d_{\text{min}}(D)}}.$$ 

So, to estimate $\delta_D$ from below, we only need to estimate $r_{\text{min}}(D)$ and $d_{\text{min}}(D)$ from below.
Theorem

We can compute a positive lower bound on $r_{\text{min}}(D)$ from $(D, \partial D, n)$.

Proof uses recent results on:

- Distortion of capacity (Thurman, 1994, [9]),
- Computation of capacity (Ransford and Rostand, 2007 [8])

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Theorem

We can compute a positive lower bound on $d_{\min}(D)$ given $(D, \partial D, n)$.

Proof uses a generalization of the Schwarz-Pick Lemma proved in 1993 by He and Schramm [3]

Theorem

From $(D, \partial D, n)$ can compute a number between $\mu_D$ and 1.

Follows from results just quoted. It now follows we can compute suitable bound on error in Koebe construction.
Corollary

Given a smooth Jordan domain $D$ and the derivatives of its boundary curves, one can compute the homeomorphic extension of $f_D$ to $\overline{D}$.

n.b. We still do not have a constructive existence proof!


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